ARITHMETIC AND THE REALITY OF NUMBERS IN THE LATE LATIN MIDDLE AGES

LA ARITMÉTICA Y LA REALIDAD DE LOS NÚMEROS EN LA BAJA EDAD MEDIA LATINA

Kamil Majcherek

Trinity College - Cambridge

Abstract

The late Middle Ages (ca. 1270-1400) in the Latin West witnessed an extraordinary rise of interest in the metaphysical status of numbers. This paper is a case study of one of the most popular arguments in favour of realism about numbers: the view according to which numbers are extramental entities distinct from the things that they number. Part one is a reconstruction of the realist argument, which is based on the commonly accepted division of sciences into real sciences and rational sciences. It is an equally commonly accepted claim that arithmetic is one of the real sciences. On the realist interpretation, for a science to be real, its object must be real. Thus, since the object of arithmetic is number, numbers must have extramental reality. Part two is an analysis of several most interesting anti-realist rebuttals of the above argument.

Keywords

Metaphysics; Medieval Philosophy; Metaphysics of Numbers

Resumen

La Baja Edad Media (ca. 1270 y 1400) fue testigo de un extraordinario aumento del interés por el carácter metafísico de los números en el Occidente latino. Este artículo es un estudio de caso de uno de los argumentos más populares a favor del realismo de los números: la idea de que los números son entidades extramentales distintas de las cosas que numeran. La primera parte es una reconstrucción del argumento realista, que se basa en la división comúnmente aceptada de las ciencias en ciencias reales y ciencias racionales. También es una afirmación comúnmente aceptada que la aritmética es una de las ciencias reales. Según la interpretación realista, para que una ciencia sea real, su objeto debe ser real. Por lo tanto, dado que el objeto de la aritmética es el número, los

números deben tener realidad extramental. La segunda parte es un análisis de varias refutaciones antirrealistas muy interesantes del argumento anterior.

Palabras clave

Metafísica; filosofía medieval; metafísica de los números

Introduction

The late Middle Ages (ca. 1270-1400) in the Latin West witnessed an extraordinary rise of interest in the metaphysical status of numbers. While the debate concerning numbers consisted in a cluster of questions, the central one throughout this period and beyond remained the same: Is a number a thing distinct from the numbered things, and if so, does it possess extramental existence?[1] Many prominent mathematicians of the twentieth and twenty-first century took a *realist* stance on this issue by giving an affirmative answer to the above question. The main rationale that one comes across over and over again is that items such as equations, proofs, and so on are not something that one invents but rather something one discovers; and that, should mathematical objects not be *real* in some sense, there would be nothing to study, or maths would prove not to be a genuine science.[2] However, there is one key difference between the way contemporary mathematicians and philosophers of mathematics understand realism about numbers and the way it was understood in the late medieval debate that I want to present and study here. Namely, while for most contemporary philosophers of mathematics endorsing realism about numbers would entail positing them as some kind of abstract objects (so that, for example, there is, properly speaking, just one number five instantiated by all numbers five used in equations etc.), for medieval thinkers debating this issue the only option on the table is a more moderate version of realism, whereby a number is an individual accident (belonging to the category of discrete quantity) of the numbered things, and each collection of numbered things has its own number that is

[1] For the summary of the debate concerning this key question, see esp. Maria Sorokina, "Numbering the Divine Persons: Mental Existence of Numbers in Duns Scotus, Henry of Harclay, and Peter Auriol", *Recherches de Théologie et Philosophie Médiévales* 87/2 (2020): 417-439; and Kamil Majcherek, "Can an Accident Inhere in More than One Subject? A Problem for Medieval Realism about Numbers", in *Pre-Modern Mathematical Thought: The Latin Discussion (13th-16th Century)*, edited by C. Crialesi (Leiden: Brill, 2025), 77–92.

[2] For just one example, see Godfrey H. Hardy, *A Mathematician's Apology* (Cambridge: Cambridge University Press, 1967), 123–124: "I believe that mathematical reality lies outside us, that our function is to discover or observe it, and that the theorems which we prove, and which we describe grandiloquently as our 'creations' are simply our notes of our observations." On mathematical Platonism, see, e.g., Øystein Linnebo, "Platonism in the Philosophy of Mathematics", in *The Stanford Encyclopedia of Philosophy*, edited by E. N. Zalta and U. Nodelman (Summer 2024 Edition), URL = <https://plato.stanford.edu/archives/sum2024/entries/platonism-mathematics/>.

numerically distinct from the number of another collection of numbered things: for instance, the number five of these five stones is an accident inhering in these five stones that is numerically distinct from the number five of these five horses.[3]

The camps in the late medieval debate can be roughly divided into three main groups. *Realists* about numbers claim that numbers are distinct from the numbered things and that they exist in the extramental reality. Many realists frame their view in hylomorphic terms, whereby the counted things are the proximate matter, i.e., the subject, of number, in which there inheres the form of a given number making the counted things be of a certain number. *Conceptualists* about numbers hold that numbers are distinct from the numbered things, but they exist not in the extramental reality but in the human mind: in the mind of the person doing the counting or thinking of a certain sum. *Reductionists* about numbers argue that numbers are not distinct from the numbered things: in their view, a number is nothing more than an aggregate of the numbered things, that is, the numbered things taken together.[4] For the purposes of this paper, I shall subsume both the conceptualists and the reductionists under the broader term *anti-realists* since the representatives of both camps deny the extramental existence of numbers.

The late medieval realists have several standard arguments in their repertoire.[5] This paper is a case study of one of them, focused on the connection between the notion of arithmetic as a *real science* and the reality of its object, numbers.[6] From among the texts that remain from our period, the argument appears in the works of authors such as the Dominican Thomist Bernard of Auvergne (d. after 1307), the Franciscan Scotist Francis of Meyronnes (ca. 1288-1328), the Carmelite Guido Terreni (ca. 1260/70-1342), and another Dominican Thomist Peter Nigri (1434-1484). It was thus ecumenical, so to speak, in that it was used by authors belonging to different religious orders and different philosophical schools; but what they all had in common was their intention to defend realism about numbers, along the lines sketched out above.

This paper shall proceed in a simple way. I shall first sketch out the usual structure of the 'arithmetic-as-real-science' argument put forward by the realists. After that, I shall

[3] For reasons why the 'Platonic' kind of realism about numbers is never seriously debated in the late Middle Ages, see, e.g., Kamil Majcherek, "Can an Accident Inhere in More than One Subject? A Problem for Medieval Realism about Numbers", in *Pre-Modern Mathematical Thought: The Latin Discussion (13th-16th Century)*, edited by C. Crialesi (Leiden: Brill, 2025), 77-92, at 95-96.

[4] For more detail on these three camps, see Sorokina, "Numbering the Divine Persons", and Majcherek, "Can an Accident Inhere", 91-96.

[5] For a case study of another standard argument from the realist repertoire, see Kamil Majcherek, "What Is It to Be Real? Numbers as Real Species of a Category in the Late Medieval Debate about the Ontological Status of Numbers", *Archiv für Geschichte der Philosophie* 107/2 (2025): 421-262.

[6] The terminology used in the texts I study in this paper is not completely uniform as some authors refer to numbers as *objects* of arithmetic whereas others call them *subjects* of arithmetic; but in both cases they mean the same thing, that is, what arithmetic investigates as a science. For the sake of clarity, in my translations, I shall render both *obiectum* and *subiectum* as 'objects' of arithmetic.

present several representative and particularly interesting anti-realist rebuttals of the argument, all but one of which consist in reformulating the notion of *real science*. Finally, I shall conclude by looking briefly at the debate from a vantage point and ask if it is possible to declare the winning party of the debate.

1. Number as an Object of a Real Science

I begin with a summary of the realist argument. The argument is based on a commonly accepted distinction between (a) *real sciences* and (b) *rational sciences*, where (a) real sciences are ones occupied with how things are in extramental reality, whereas (b) rational sciences, of which logic is a prime example, are concerned with our concepts and thinking more generally. The key assumption of the argument, which the anti-realists usually do not challenge, is that:

1. Arithmetic is a real science.

Arithmetic is a real science because it is ultimately concerned with how things are in extramental states of affairs: for example, that there are five stones in that pile, or that there are twelve people in the classroom.[7]

The key, and very contentious, move that the realists then need is to argue that:

2. If numbers (including their forms) did not have extramental existence, arithmetic would not be a real science.

As some of the realists make clear, this inference holds because a science derives its reality from the reality of its object: that is to say, that science is real whose proper object is real. For example, Guido Terreni says that:

3. A science is called *real* thanks to the reality of its object.[8]

Peter Nigri makes the same point in a similar fashion by stating that:

3*. No subject of a real science is made by the soul [...] because a science is called *real* by extrinsic denomination thanks to <its> real object.[9]

[7] As said above, this will no doubt come across as counterintuitive to many contemporary readers, but it has to be borne in mind that the medievals are focused on the status of numbers *in concreto*, that is, as individual accidents of (groups of) individual subjects, rather than as numbers *in abstracto*, that is, as universals.

[8] Guido Terreni, *Quodlibet* (henceforth *Quodl.*) I, q. 8 (ms. Vatican, Biblioteca Apostolica, cod. Borgh. 39, 14r-241v), 26va: "Scientia dicitur realis a realitate sui subiecti."

[9] Peter Nigri, *Clypeus Thomistarum* (ms. Munich, Bayerische Staatsbibliothek, cod. Clm 26722), f. 102va; (ed. Venice 1481, 102va): "Nullum subiectum scientiae realis est factum ab anima. Sed numerus qui est quantitas discreta est subiectum scientiae realis. Igitur etc. Maior nota, quia scientia dicitur [...] realis ab obiecto reali." See also, e.g., how this argument is summarised by Francis of Marchia in *Scriptum in I Sententiarum* (henceforth *Scriptum*), d. 24, a. 3 (unpublished

Given that the proper object[10] of arithmetic is number, if arithmetic is to be a real science, it must be the case that:

4. Numbers have extramental existence.

And because, as we know, numbers are conceived by most realists as hylomorphic composites, and because on this understanding what makes each number a number and what makes a number a number of a given kind is its form, it is necessary that it is not only the *matter* of the number but also the *form* of the number that has extramental existence. This is made explicit by, for example, Bernard of Auvergne, who concludes from the above that:

5. Therefore, it is necessary that a number exist outside <the soul> not only in respect of its unities which are its matter but also in respect of its form.[11]

2. Anti-Realist Critique of the Argument and Realist Reply

2.1. *Henry of Harclay. Reformulating the Reality Criterion*

I now turn to the anti-realist camp. I begin with the Franciscan Henry of Harclay (1270-1317). In his *Sentences* commentary, Henry formulates the realist argument in the following way, narrowing it down to three *scientiae reales*:

> There are three real sciences, as is said in Book VI of the *Metaphysics*: natural science, mathematical science, and the science of the divine. But arithmetic is the first of

edition by G. Etzkorn), 40: "Item, obiectum scientiae realis est reale, *quia scientia specificatur ex obiecto*. Sed numerus est per se subiectum arithmeticae, quae est scientia realis; ergo etc." (italics mine). For other examples of this argument, see, e.g., Bernard of Auvergne, *Reprobationes Henrici de Gandavo Quodlibet* (henceforth *Reprobationes Henrici*) IV, q. 6 (ms. Bologna, Biblioteca Communale dell'Archiginnasio, cod. A.943), 21ra: "Forma denarii est aliquid extra intellectum, [...]. Nisi enim esset aliquid extra intellectum, [...] arithmetica non esset realis scientia. Unde oportet quod numerus sit ab extra non solum secundum unitates materialiter se habentes sed etiam secundum formam"; Terreni, *Quodl.* I, q. 8, 26va: "Scientiae reali oportet dare obiectum reale, quia scientia dicitur realis a realitate obiecti. Sed arithmetica est scientia realis, sexto *Metaphysicae*, cuius obiectum est numerus. Ergo numerus est ens reale"; Francis of Meyronnes, *In I Sententiarum (Conflatus)*, d. 24, a. 1 (ed. Venice 1520), 79rb: "Illud quod est obiectum scientiae realis est reale. Sed numerus est huismodi, quia est obiectum arithmeticae. Maior est communiter concessa. Ergo etc."

[10] I here use the terms 'subject of science' and 'object of science' interchangeably because the later medieval philosophers did so themselves; the *subject* or *object* of a given science is what that given science investigates per se.

[11] See Bernard of Auvergne, *Reprobationes Henrici* IV, q. 6, 21ra: "Sic ergo patet quod numerus est aliquid extra animam, non solum quantum ad materiam sed etiam quantum ad formam."

> mathematical sciences. Therefore, this science is real, and hence so its subject, too, is a real being outside the soul. But its subject is number. Therefore etc.[12]

In response to this objection Henry puts forward a reformulation of what it means for a science to be real. Recall that for the realists, a science derives its reality from its object, so that for a science to be real its object must first be real as well. According to Henry, real sciences study not individual real beings but rather *primary intentions*: the concepts under which these real beings fall. A science is called *real* as long as it is occupied with primary intentions, whereas it is a rational science if it studies *secondary intentions*, which, as Henry puts it, are "produced by the intellect in the soul" (*fabricantur ab intellectu in anima*).[13] Sciences are general, so they need concepts (encompassing the individuals of a given species) to work on. Since these primary intentions are acquired by abstraction from individual, real, extramental entities, in this case, numbered things, there is (provided the abstraction is conducted in the right way) no fiction or distortion in the coming to be of their concepts, and this is why the sciences operating on them are called *real*. Indeed, Henry holds that fiction or deception would be involved if a number were depicted as differing from the numbered things in reality since, according to Henry (for reasons that I shall not explore here), a number is distinct from the numbered things not in reality but only in the intellect.[14]

2.2. *Peter Auriol. Science as Working on Universals*

Another Franciscan, Peter Auriol (ca. 1280-1322), follows Henry's account quite closely, for example, by holding that "for something to be an object of a real science it suffices that it [i.e., the subject] be a primary intention". But Auriol also adds some new considerations over and above those laid out in Henry's text. He emphasises that every science is occupied with universals, not individuals (a point inspired by Henry). Thus, if

[12] See Henry of Harclay, *In I Sententiarum,* q. 48 (d. 24) (henceforth *In I Sent.* 48), edited by M. Sorokina, "Henri de Harclay sur l'ontologie des nombres: à l'origine d'un désaccord entre Pierre Auriol et Thomas Wylton", *Archives d'histoire doctrinale et littéraire du Moyen Age* 89 (2022): 35-94, 90: "Praeterea, tres sunt scientiae reales, sexto *Metaphysicae*: naturalis, mathematica et divina. Sed arithmetrica est prima scientiarum mathematicarum. Igitur illa scientia est realis; ergo et eius subiectum est ens reale extra animam. Sed eius subiectum est numerus; ergo etc." See Aristotle, *Metaphysica*, lib. VI, c. 1, 1026a18-19.

[13] There is an extensive body of literature on intentionality in the late Middle Ages, though not on Harclay specifically. For a starting point, see, e.g., Dominik Perler, *Theorien der Intentionalität im Mittelalter* (Frankfurt am Main: Vittorio Klostermann, 2020).

[14] Henry of Harclay, *In I Sent.* 48, 92: "Ad secundum dico quod arithmetica est realis scientia, quia non considerat res secundae intentionis, quae fabricantur ab intellectu in anima. Sed nihil prohibet artificem realem considerare rem primae intentionis, quae est in anima facta a re extra. Sic dico quod geometriae subiectum est ens solum in anima sub illa ratione qua est subiectum geometriae; quia quantitas abstracta non habet esse nisi in intellectu, aliter enim abstractum esset mendacium. Ita dico quod numerus in re non est aliud a rebus numeratis, sed ut distinguitur a rebus numeratis habet tantum esse in intellectu." See Sorokina, "Henri de Harclay", 48.

universals were not identical with the concepts considered by the scientists, universals would need to be posited in reality; a conclusion that Auriol finds unacceptable.[15] (Needless to say, he would face a fierce opposition from many more realistically minded contemporaries and posteriors on this point).

As has been said, Auriol's point of departure seems to be the thesis that a science is occupied not with individuals but with universals—an assumption most of his contemporaries would share (albeit with different thinkers understanding it in very different ways, depending on their metaphysical commitments regarding universals, and also their view on the object of science). In Auriol's case, the reasoning seems to be that, since a science works on universals, the universals need to be posited somewhere. If they are not posited in the mind, they have to be posited in reality (Auriol seems to treat this as an exhaustive alternative); and since, according to Auriol, there are no universals in reality, they must be posited in the mind, as its primary intentions. The reason why a science must be posited to be occupied with universals is simple: the conclusions of each science must be general; that is to say, each science needs to make a conclusion not about just one or a few individuals of a given species but about all individuals of that species, which means working on universals rather than individuals or any sets smaller than entire genera.

Auriol clearly seems to assume that if he can show that the properties of arithmetic do not exist in extramental actuality, or at least do not need to so exist for arithmetic to be a real science, then it will be shown that the reality of arithmetic does not depend on the reality of its object in the way conceived by the realists. Auriol proceeds by drawing an analogy with the properties studied by the geometer, which he also holds do not need extramental existence. It is not fully clear, though, how Auriol's reasoning unfolds. Auriol tells us that "the equality of three angles to two right angles is not in actuality outside the intellect since the two are not always right in actuality when the triangle exists". What he seems to mean is that what is always there are only the three angles but not the two angles being right; as a result, it is only the human intellect that can establish that the triangle's angles are equal to its two right angles. From this, in turn, it follows that the property of equivalence, which is an instance of a primary intention, does not exist in the extramental reality. This is so because this property is a relation, and a relation cannot have actual existence if one of its terms does not actually exist, as is the case here.[16] Auriol

[15] On Auriol on universals, see, e.g., Francis E. Kelley, "Walter Chatton vs. Aureoli and Ockham Regarding the Universal Concept", *Franciscan Studies* 41 (1984): 222-49, and Christian Rode, "Peter Auriol on Universals and the Notion of Passive Conception", in *Universals in the Fourteenth Century*, edited by F. Amerini and L. Cesalli (Pisa: Edizioni della Normale, 2017), 139-154. On Auriol on intentionality, see, e.g., Russell L. Friedman, "Peter Auriol on Intentions and Essential Predication", in *Medieval Analyses in Language and Cognition*, edited by S. Ebbesen and R. L. Friedman (Copenhagen: The Royal Danish Academy of Sciences and Letters, 1999), 415-430.

[16] See Peter Auriol, *Scriptum in I Sententiarum* (henceforth *Scriptum*), d. 24, a. 1 (unpublished edition by M. Sorokina), 17: "Non valet etiam octavum, quia subiectum scientiae realis non oportet esse realiter extra, prout subicitur passionibus quae inquiruntur in scientia; alioquin sequeretur quod

then extends the same point to the case of properties studied by the arithmetician, such as the property of being a square of; such a property too has no actual existence; rather, it only exists in the soul. While this remains implicit here, it is clear on the basis of what has been said above that Auriol infers from this that arithmetic does not need the actual existence of its object to be a real science, as long as it operates based on universals abstracted from extramental individuals. The arithmetician brings numbers and their properties out of the potency of extramental individuals, as it were, but the actuality that she imposes on them only has mental existence.[17]

If this interpretation of Auriol's somewhat elliptical statements is correct, then it seems quite evident to me that his opponent would remain unimpressed. If we take Auriol's example of being the square of, Auriol does not give us any proof for why this property cannot have actual extramental existence; and clearly to the realist the opposite would be obvious, that since there are extramental numbers, they also stand in certain extramental relationships to each other, and one of them is one number being the square of another. If we go one step back and look at Auriol's geometrical example, the only rationale that he seems to give us for why the property of equivalence cannot have extramental existence is that a triangle can exist while its two angles may not always be right. But I do not see how that would convince, or even bother, the realist: after all, the number of, say, the sheep in a given flock can fluctuate, but the realist does not think that this is any evidence that number itself has no extramental existence. Indeed, this seems to be the murkiest moment in Auriol's reasoning, since, more generally speaking, I cannot see why the fact that a certain property does not always obtain while its bearer, or bearers, do would constitute evidence that this property does not have extramental existence. I must therefore leave it to the reader to decide, also on whether my reading of Auriol is correct.

2.3. *Francis of Marchia. Arithmetic and Other Real Sciences*

I now turn to another great Franciscan, Francis of Marchia (ca. 1285/90-after 1344), whose theory builds upon both Henry of Harclay and Peter Auriol. Francis's reply to the

universalia essent in rerum natura, cum de universalibus sit omnis scientia et non de singularibus, ut dicitur septimo *Metaphysicae*. Sufficit ergo ad hoc quod aliquid sit subiectum scientiae realis quod sit intentio prima. Et confirmatur ex passionibus quas geometra inquirit aut arithmeticus. Non enim aequalitas trium angulorum ad duos rectos est in actu extra intellectum, cum nec duo recti semper sint in actu dum est triangulus. Nulla autem relatio est in actu termino non existente."

[17] See Peter Auriol, *Scriptum*, 17: "Nulla autem relatio est in actu termino non existente. Similiter quadratura et superficialitas aut esse cubitum, quae circa numeros considerat arithmeticus, non habent esse nisi per apprehensionem intellectus numeros comparantis ad continuas quantitates. Sic igitur numerus pertinet ad scientiam realem, quia res extra sunt in potentia ad formam numeri, sicut res particulares sunt in potentia ut abstrahantur inde universalia de quibus est omnis scientia subiective."

present objection is in fact extraordinarily long and detailed. What distinguishes him somewhat from the other two thinkers is his focus on comparing arithmetic to metaphysics and physics; regarding all of these he clearly assumes that they are real sciences. Francis begins by stating openly that arithmetic, geometry, metaphysics, and physics (to which he also later adds moral sciences) are all disciplines that are "essentially concerned with being in the soul" rather than extramental being.

According to Francis, this is clear with regards to (a) metaphysics (understood as *scientia divina*), which has as its object the causality of the First Cause; Francis holds that this causality "is formally a being of reason, because the relation of each cause and the relation of the principle and the relation of eminence and the relation of first priority towards beings are relations of reason". These claims seem to be based on the commonly accepted claim that there is no real relation between God and creatures going from God towards creatures; rather, all such relations are, as Francis himself says, relations of reason.[18] Thus, metaphysics is a real science despite the fact that its object is a being of reason. (b) Physics, according to Francis, is primarily occupied with the First Mover. But, for the same reason (that is, because there is no real relation from God towards the creature), the relation of the First Mover towards what is moved is a relation of reason too; so, the object of physics is a being of reason. (c) Geometry considers measures such as being two cubits long etc., which, according to Francis, are relations of reason as well.[19] (d) Finally, Francis even goes on to add the moral science, which, he says, is a real science too (since it is concerned with extramental reality): its object are things such as agreements, deposits, and purchases, which are all mere relations of reason—hence the same conclusion as before follows.[20]

[18] The main reason for the denial that the relations going from God towards creatures are real is the worry that this would undermine the divine simplicity. On medieval theories of relations, see, e.g., Mark Henninger, *Relations: Medieval Theories 1250-1325* (Oxford: Oxford University Press, 1989).

[19] It is an interesting question, which has been brought to my attention by the anonymous referee, why Francis holds that being of a given dimension, e.g., being two cubits long, is a relation of reason. One possible interpretation, also suggested by the referee, with whom I am inclined to agree, is that this is because measuring involves the application of a certain unit of measure, and as such it is an activity performed by the (human) mind; which makes the dimension or measure itself a mere being of reason.

[20] See Francis of Marchia, *Scriptum*, bk. I, d. 24, a. 1, 48-49: "Ad secundam rationem dico quod arithmetica non est magis scientia realis quam metaphysica et physica et geometria, et quaelibet istarum est per se de ente in anima. De metaphysica patet, quia est de causalitate primae causae, cuius causalitas est ens rationis formaliter, quia relatio causae cuiuscumque et relatio principii et relatio eminentiae et relatio primitatis in ente primo ad alia entia est relatio rationis; et tota metaphysica est principaliter de hoc. Similiter in physica principaliter tractatur de primo motore; relatio autem motoris primi ad mobile est relatio rationis. Similiter geometria per se considerat mensuras bicubiti, tricubiti et de aliis, quae sunt relationes rationis. Similiter scientia moralis, quae est scientia realis, per se considerat pactiones, contractus, pignora, mutua,

In light of this enumeration of real sciences having as their objects *entia rationis*, Francis proposes his own reformulation of the criteria of what counts as a real science. He states that there are two possibilities. Firstly, a science can be called 'real' for the simple reason that it concerns real being. (Given how exhaustive Francis's enumeration above was, however, I find it difficult to think of a science that would fulfil this criterion, since regarding each of the real sciences listed above Francis claimed that their objects are *entia rationis*.) The second possibility is that a science is called 'real' "because it concerns a being of reason in a real being"; that is to say, it concerns a being of reason insofar as it is founded upon real being and not upon another being of reason detached from real being. (This is in contrast with a rational science, which studies beings of reason as abstracted from all real being; for instance, "logic concerns the second intentions abstracted from all primary intentions".) All of the real sciences listed above, Francis seems to believe, satisfy this latter criterion, that is, studying beings of reason as founded upon real beings. For example, the moral sciences study the relations of reason such as selling or making contracts; but the foundations of these relations of reason are people entering contracts or selling things, who are real beings.[21]

This is not the end of the story, however, since Francis's opponent, be that real or imaginary, holds his ground and objects to what has been said by stressing that the above considerations do not suffice to make arithmetic a real science, because "arithmetic concerns a number in an absolute way, as abstracting from all matter, not as concerning determinate matter". From this it follows, against Francis, that arithmetic is not a real science because the connection between the being of reason (number) and the real being (numbered thing) is severed.[22]

Quite extraordinarily, Francis contemplates three different replies to this further objection. The first possible reply begins with a distinction between two different kinds of being of reason. Some of them are second intentions (concepts which concern other concepts, e.g., *genus*); they are the subject matter of sciences such as logic and grammar, which are thus not real sciences. Others, on the other hand, are primary intentions (concepts which concern things outside the mind, e.g., *human*); even though they are objectively in the soul, they are based on extramental real beings that the soul grasps primarily and directly rather than reflexively and discursively. The latter, thanks to the

deposita, emptiones et venditiones, quae omnia dicuntur relationes rationis, et tamen est scientia realis."

[21] See Francis of Marchia, *Scriptum*, bk. I, d. 24, a. 1, 49: "Ideo dico quod scientia realis dicitur vel quia est de ente reali, vel quia est de ente rationis in ente reali, inquantum fundatur super ens reale, et non est de ente rationis ut abstrahit ab ente reali, sicut sunt omnes scientiae praedictae. Scientia vero rationalis dicitur quia est de entibus rationis ut abstrahit ab ente reali, sicut logica est de intentionibus secundis abstractis ab omnibus intentionibus primis."

[22] See Francis of Marchia, *Scriptum*, bk. I, d. 24, a. 1, 49: "Sed illud non videtur sufficere, quia arithmetica est de numero absolute, ut abstrahit a qualibet materia, non ut concernit materiam determinatam. Si ergo numerus est ens rationis, sequitur quod arithmetica non erit scientia realis."

connection to real beings that I have just described, can be the proper object of a real science.[23] This reply is closely connected to what we have seen Francis say above about the *entia rationis* as founded upon real being, but it also nuances his claim and further elaborates upon it.

The second possible reply to the realist rejoinder that Francis contemplates is brief: It could be said that when the opponent speaks of an abstract number, he speaks of a number that, even though it belongs to the genus of discrete quantity, is nevertheless not the object of arithmetic but rather belongs to metaphysics, because of its perfectly abstract status. By contrast, the number considered by the arithmetician is a material number present in material things that have quantity.[24] As before, making the distinction and pointing to the second kind of number as one studied by arithmetic secures, in Francis's view, the status of arithmetic as a real science because it preserves its connection to real being.

Incidentally, this is another opportunity to point out the peculiar (from a contemporary point of view) late medieval approach to studying the metaphysical status of numbers: the Latin philosophers in 1250-1400 are much more interested in the latter kind of number, that is, numbers as present in material things, and it is concerning their status that they conduct all the debates analysed here; whereas the first kind of number, the abstract number, never gets singled out as a topic for a special debate, and if anything, seems to simply be subsumed under the general discussion concerning universals.[25]

Finally, the third possible reply contemplated by Francis is based on a fourfold order of being that he lays out. The order is a matrix based on two criteria. The first criterion is whether a given being *is objectively in the soul*. To be objectively in the soul means to be thought of by the soul but in such a way that a given thing would not exist without being in the soul; nevertheless, the soul does not make up its object; rather, the object of thinking is based on how things are in reality. The second criterion is whether a given being *is effectively from the soul*. Unlike the first criterion, where the soul thinks of something that is, at least in a sense, independent of it, in that the soul does not make it

[23] See Francis of Marchia, *Scriptum*, bk. I, d. 24, a. 1, 49: "Potest dici quod entia rationis sunt in duplici differentia: quaedam sunt intentiones secundae, et de talibus est logica et grammatica; ideo non dicuntur proprie scientiae reales. Quaedam sunt intentiones primae; et talia, licet sint tantum in anima obiective, tamen se habent ad actum animae sicut intentiones extra, quia apprehenduntur in prima apprehensione et directe, non per reflexionem. Et quantum ad hoc scientia de eis potest dici scientia realis, quia est de obiectis constitutis per intellectum quasi modo naturae, absque discursu et collatione."

[24] See Francis of Marchia, *Scriptum*, bk. I, d. 24, a. 1, 49-50: "Aliter potest dici quod numerus abstractus ab omni materia, licet sit quantitas discreta, non tamen pertinet per se ad arithmeticam, sed sic abstractus numerus pertinet ad metaphysicum; ad arithmeticam vero pertinet consideratio numeri materialis in rebus habentibus quantitatem, et sic arithmeticus est artifex realis."

[25] For a somewhat more detailed attempt at explaining this late medieval assumption, see Majcherek, "Can an Accident Inhere", 95.

up, if something comes from the soul effectively, it is produced by the soul and thus entirely dependent in what it is on the soul.[26]

With these preliminaries in mind, we can examine Francis's fourfold order of being. (1) First, there are beings that are neither objectively in the soul nor effectively from the soul. An example that Francis gives are natural beings, which are a given to our intellects: they have entirely extramental existence (although, of course, they can be grasped by our intellects: it is just that in their essence and existence they do not depend on being thought of by our intellects). (2) Second, there are beings that are both objectively in, and effectively from, the intellect only: for example, all second intentions fall under this category: e.g., genus, individuum, proposition, and syllogism. All of these exist only in (and so depend for their existence on) the mind and are produced by the mind. (3) Third, there are beings that are outside the mind, and are thus like natural beings, but are effectively from the mind, thus being in this respect closer to primary intentions. Francis's example are artefacts: they are effectively from the intellect of the artificer but once they have been produced, they are independent of the intellect. Finally, (4) fourth, some beings are objectively in the soul only but are not effectively from the soul unless, as Francis puts it, it be "indirectly and accidentally". It is under this category that numbers fall. They exist as objects of thinking only but are not in any way conjured up by the soul.[27]

[26] Francis's third solution, and its reliance on the distinction between being effectively by the soul vs. being objectively in the soul, was first pointed out by William O. Duba, "Three Franciscan Metaphysicians after Scotus: Antonius Andreae, Francis of Marchia, and Nicholas Bonet", in *A Companion to the Latin Medieval Commentaries on Aristotle's Metaphysics*, edited by F. Amerini and G. Galluzzo (Leiden-Boston: Brill, 2014), 413-493, at 458. I am indebted to the anonymous referee for this reference. For more on Francis of Marchia on intentionality, see, e.g., William O. Duba, "Neither First, nor Second, nor... in his Commentary on the *Sentences*. Francis of Marchia's *intentiones neutrae*", *Quaestio* 10: 285-313.

[27] See Francis of Marchia, *Scriptum*, bk. I, d. 24, a. 1, 50: "Aliter potest dici ad istam rationem et ad omnes sequentes quod licet numerus sit tantum in anima obiective, non tamen est per se ab anima effective. Unde potest poni quadruplex ordo entium. Quaedam sunt entia quae non sunt ab anima effective, nec sunt tantum in anima obiective, sicut sunt entia naturalia, quae non dependent ab anima obiective nec effective. Quaedam vero entia sunt quae sunt tantum in anima obiective et per se ab anima effective, sicut sunt secundae intentiones omnes, ut intentio generis et individui, propositionis, syllogismi, quae non sunt nisi tantum in anima obiective et ab anima effective. Quaedam vero entia sunt quae sunt tantum ab anima effective et sunt extra animam subiective sicut sunt entia artificialia, quae sunt ab intellectu effective et non a natura, sed sunt extra subiective. Quaedam vero entia sunt quae sunt tantum in anima obiective, sed non sunt per se ab anima effective nisi tantum indirecte per accidens, sicut sunt omnes species numerorum quae, licet sint tantum obiective in anima, non tamen sunt per se effective ab anima, sed quaelibet earum consequitur unitates quodam ordine conceptas, et sicut res extra consequuntur proprie accidentia extra, ita unitates quodam ordine conceptas consequitur propria forma numeri secundum diversam proportionem unitatum conceptarum quae non consequitur eas extra, quia unitates extra distinguuntur subiecto et loco; eadem autem forma accidentalis non potest simul fundari in diversis subiectis loco et subiecto distinctis. Sed unitates conceptae apud intellectum

However, the way in which Francis's third reply proceeds seems to put the connection between numbers and reality at jeopardy, since he seems to deny any relation or correspondence between the order of unities within a number and the order of extramental unities. Francis begins on a note that could seem to suggest the opposite: he again reiterates that numbers are not effectively from the soul but are only in it objectively. Then he adds that each number "follows unities conceived in a certain order", and the unities conceived in their order "are followed by the proper form of number in accordance with the diverse proportion of conceived unities". (This, then, is an example of an emergentist theory of the form of number, whereby the form arises from a given order of unities.)

Then, however, Francis adds that the diverse proportion of unities in the mind does not correspond to, or follow from, how things are outside the intellect. The reason that he gives, which I cannot discuss here in detail, is one of the standard arguments against realism about numbers: one form needs one subject; but the unities that would be the subject of the extramental form of number are not unified into a single subject since they are themselves distinct in subject and can also be distinct in place.[28] This problem obviously does not plague unities present in the intellect, since these are "distinct neither in place nor in subject". For Francis this implies not only that the form of number has no extramental existence but also that the form of number arises not from the unities in the extramental reality but from the unities as conceived by the mind. At this point it looks almost as if Francis has noticed that his considerations have cast serious doubt onto the reality of the object of arithmetic and hence onto the status of arithmetic as a real science. As a last-ditch attempt (which, to my mind, is entirely unconvincing), he returns to the distinction between rational and real sciences by saying that:

> And thus this is the difference between the second intentions considered by logic and the primary intentions considered by arithmetic, because logic concerns second intentions, which are only in the soul and from the soul, and for this reason it is not a real science; but arithmetic concerns primary intentions, which are not from the soul but follow from things according to the mode of being in the soul; and in that respect arithmetic can be called real.[29]

non distinguuntur loco nec subiecto. Ideo forma numeri consequitur eas apud intellectum et non consequitur eas in re extra. Et tunc haec est differentia inter intentiones secundas quas considerat logica et intentiones primas quas considerat arithmetica, quia logica est de intentionibus secundis quae sunt tantum in anima et ab anima, ideo non est scientia realis; sed arithmetica est de intentionibus primis quae non sunt ab anima sed consequuntur res secundum modum essendi in anima, et quantum ad hoc potest dici realis."

[28] For the text, see n. 25. For a discussion of this problem, see Kamil Majcherek, "Can an Accident Inhere".

[29] Francis of Marchia, *Scriptum*, bk. I, d. 24, a. 1, 51: "Et tunc haec est differentia inter intentiones secundas quas considerat logica et intentiones primas quas considerat arithmetica, quia logica est de intentionibus secundis, quae sunt tantum in anima et ab anima, ideo non est scientia realis;

As I have said, I find this explanation unconvincing. In his third reply, Francis has divorced numbers so far from the state of reality that he can no longer reconnect the two, and ends on a very weak note: arithmetic can be called *real science* in some respect, namely, insofar as it considers primary intentions; as opposed to rational sciences, which consider second intentions. But, given the strong divide between arithmetic and reality, one might have serious doubts about whether this really suffices to make arithmetic a real science; this is, I take it, the source of Francis's qualification that arithmetic can be called *real* in a certain respect only.

2.4. *William of Ockham. Rejection of Real Science*

William of Ockham famously defends an original view of science as an aggregate of pieces of knowledge.[30] This, however, has little influence on what he says about the realist argument I am presently discussing. Ockham denies that "sciences have real distinct subjects"—rather, in his view, they are occupied with concepts. Thus, the only element of *reality* involved in a science are the concepts in my mind, which are real qualities (habits of my mind; Ockham is a realist about several kinds of quality), in the sense that they are distinct from my mind but obviously not in the sense that they exist outside the mind.[31] Therefore, Ockham's case involves not, as in the authors discussed above, an attempt to accommodate the assumption about the reality of the science of arithmetic but rather the rejection of this assumption; for this reason, I am only mentioning him very briefly here.

Ockham refuses to take the talk of *real objects* and *sciences* seriously. One of the examples he gives is this. I can certainly gain knowledge (*scientia*) of people. People exist outside the mind, so the science concerning them that gets developed would need to count as a real science. Yet it is clear that "people do not signify some one thing really and totally distinct from each human being, nor one <thing> composed of them." According to the realists, each science has its own proper subject. But the science of the people is different from the science of an individual human being because it proves different properties of its subject. Furthermore, the science of the people should, as I said, count as a real science, and yet its subject is *not* real because there is no such thing, Ockham says, as people—there is just this individual human being, that individual human

sed arithmetica est de intentionibus primis, quae non sunt ab anima, sed consequuntur res secundum modum essendi in anima, et quantum ad hoc potest dici realis".

[30] On Ockham's theory of *scientia*, see Jenny Pelletier, *William of Ockham on Metaphysics: The Science of Being and God* (Leiden-Boston: Brill, 2012), Chapter 1.

[31] See Ockham, *Ordinatio in I Sententiarum* (henceforth *Ordinatio*), d. 24, a. 2, in *Guillelmi de Ockham Scriptum in librum primum Sententiarum. Distinctiones XIX–XLVIII*, edited by G. Etzkorn and F. Kelley, vol. IV of *Guillelmi de Ockham Opera Theologica*, edited by G. Gál et al. (St. Bonaventure, NY: Franciscan Institute, 2000), 115: "Numquam scientiae de quibus loquitur Philosophus habent subiecta realia distincta, nisi ponatur quod conceptus sit qualitas subiective in anima." On Ockham's realism about quality, see, e.g., Marilyn McCord Adams, *William of Ockham* (Notre Dame, IN: University of Notre Dame Press, 1987), 277-286.

being, and so on. Claiming that the science of the people is somehow just a science of a human being taken in the plural will not do, since, as I said, the properties proven by the science of the human being differ from the properties proven by the science of the people, so the two must be distinct. The only concession that Ockham is ultimately willing to make is to admit that arithmetic (and each other science) has one subject matter, because the concept of number is one and distinct from others.[32] A reply somewhat similar to Ockham's was also given by another reductionist about numbers, William of Rubio. Rubio's purpose in his response to the argument is to show that it is not necessary that when one science is really distinct from another, their subjects must be really distinct too. In a nutshell, his main point is that the same thing, or things, can be considered from different perspectives in virtue of the fact that one and the same thing "can have both diverse acts and diverse habits". For example, I derive a different concept from whiteness when I consider it absolutely, that is, in itself, and a different concept when I compare it to another whiteness and thus grasp its similarity to the other whiteness. Thus, I can also have a different concept of things taken together as numbered together and a different concept of each of them taken separately.[33]

3. Conclusion

Is it possible to declare a clear winner in the debate I have reconstructed and analysed above? It seems to me that it is not: the two camps employ two very different criteria of the reality of science; for the realists, the reality of science depends on the reality of its object in the sense that the object of a real science must itself have extramental reality, whereas for the anti-realists, the reality of science depends on the reality of its object in a different sense, whereby the object, as a primary intention, is based on how things are

[32] See Ockham, *Ordinatio*, lib. I, d. 24, a. 2, 115-16: "Ad tertium diceretur quod numquam scientiae de quibus loquitur Philosophus habent subiecta realia distincta, nisi ponatur quod conceptus sit qualitas subiective in anima. Nec etiam semper subiecta distincta distinctarum scientiarum supponunt vel stant pro rebus realiter et totaliter distinctis. Unde alia scientia potest esse de homine et de populo. Unde de homine possum scire quod homo est risibilis, quod est susceptibilis disciplinae, quod est beatificabilis, et sic de aliis. De populo possum scire quod exercitus debet eligere ducem vel principem, quod debet unanimiter congredi contra inimicos, et sic de aliis conclusionibus quae sunt distinctae scientiae. Et tamen exercitus vel populus non significat aliquam unam rem realiter et totaliter distinctam a quolibet homine, nec etiam unum compositum ex eis. [...]. Et si dicatur quod unius scientiae est unum subiectum, concedendum est quod est unum subiectum, quia iste conceptus 'numerus' est conceptus unus et distinctus conceptus."

[33] See Rubio, *In I Sententiarum*, d. 24, a. 2 (ed. Paris 1518), 168va-b: "Non omnes scientiae realiter distinctae habent subiecta in re extra distincta, cum eiusdem rei possunt esse tam actus quam habitus specie diversi. Sicut enim alium conceptum habeo de una albedine quando eam intelligo absolute, et alium quando comparo ipsam ad aliam intelligendo quod est talis qualis illa, et per consequens ipsi similis, ita consimiliter alium conceptum et habitum habere possum de pluribus rebus simul sumptis ab earum numero distinctis, et alium de quaelibet ipsarum tantummodo per se sumptis. Quare etc."

in the extramental reality, but the object itself does not have to possess extramental reality. The two sides are hence able to sustain their very different conceptions of the metaphysical status of number against each other's objections. It is worth emphasising that these two different understandings of reality of science are by no means an *ad hoc* manoeuvre used by the two sides; rather, they are considered views of our authors on what an object of science and what the reality of science consist in, the articulation of which was necessitated by the realist argument tying the reality of number to the reality of the science of arithmetic. This, however, leads me to my final conclusion, which is that in the debate at hand, both sides seem to a large degree to be speaking past each other: what is at stake is the extramental existence of numbers, but what informs the arguments and rebuttals used in the debate about this issue is the set of background assumptions concerning the reality of science, which sets the two camps so far apart that one can have serious doubts about whether these arguments and rebuttals ever reach their targets.

Kamil Majcherek
km713@cam.ac.uk

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